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Ward identities and characters of Kac–Moody algebras

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Abstract. We derive partial differential equations for characters of Kac–Moody algebras by using Ward identities for energy-momentum tensors and currents on a torus and the null vectors of current algebras. The solutions of the partial differential equations for affine algebras $A_l^{(1)}$, $D_l^{(1)}$, $E_l^{(1)}$ ($l=6,7,8$) and $B_l^{(1)}$, at level one, are given.

1. Introduction

Two-dimensional conformal field theories have been extensively studied [1], both for their purely mathematical interest and for their applications to critical phenomena and string theories. An important variety of conformal field theories is Wess–Zumino–Witten (WZW) models [2, 3]. These models on higher genus Riemann surfaces provide an elegant example for examining interacting conformal field theories. As conformal field theories, WZW models possess the conformal symmetry that leads to the existence of Ward identities for energy-momentum tensors and implies that the modes of the energy-momentum tensor form a Virasoro algebra. Furthermore, in WZW models there exists, in addition to conformal symmetry, group or Kac–Moody symmetry that leads to the existence of Ward identities for currents and implies that the modes of the currents form a Kac–Moody algebra. One obtains mixed Ward identities and combined Virasoro and Kac–Moody algebra by using the Sugawara construction. The Hilbert space of a theory is decomposed into a finite sum of irreducible representations of the algebra for a given level and the modular invariant one-loop partition function can be obtained from the characters of the Kac–Moody algebra [3].

Ward identities on higher genus Riemann surfaces for both energy-momentum tensors and currents have been given in [4, 5]. However, in the current Ward identities the existence of the terms which contain zero modes of currents in the correlation functions in fact makes the Ward identities incomplete or weak. In the case of the torus, Bernard [6] introduced expectation values by the insertion of an element of the group G and derived the complete current Ward identities.

A new proof of the Weyl–Kac character formula has been given by Bernard in [6]. He derived a heat equation by using the Ward identities and proved by algebraic methods (using the affine Weyl group) that the solution of the heat equation is just the character given in the Weyl–Kac character formula. Recently, Eguchi and Ooguri

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gave a different proof for group SU_2 and their proof is 'completely physical' in the sense that only physical Ward identities and null vector fields are used [7].

In this paper, we generalize their method to the general simple Lie groups. Our purpose is twofold. One is to prove the Weyl–Kac character formula for general cases in the 'completely physical' way. The other is to provide a new way of calculating string functions. The calculation of string functions is important in order to obtain explicit expressions for the one-loop partition functions for wzw theories. In many cases string functions can be easily found by using this method (analytical method). We derive partial differential equations satisfied by characters of affine Lie algebras in section 2. In section 3, we solve the partial differential equations for level one. In section 4, we show how to calculate string functions by using the analytical method and give some concrete examples. Finally, some conclusions are drawn and discussed in the last section.

2. Partial differential equations for characters of Kac–Moody algebras

In the wzw model for simple Lie group G, the modes of the currents form an untwisted affine algebra [3]

$$
\begin{aligned}
[J^i_m, J^j_n] &= C^{ij}_a J^a_{m+n} + Km\delta_{m,-n}\delta^{ij} \\
[J^i_m, J^s_n] &= C^{is}_a J^a_{m+n} \\
[J^r_m, J^s_n] &= C^{rs}_a J^a_{m+n} + Km\delta_{m,-n}\delta^{r,-s}
\end{aligned}
\qquad (2.1)
$$

where we have assumed the Cartan–Weyl bases for the finite algebra G of the group G (we denote the group and its algebra by a same letter G whenever no confusion arises) so that

$$
\begin{aligned}
&C^{ij}_a = 0 \qquad C^{is}_a = \delta_{as}\alpha^{(s)}_i \qquad C^{rs}_i = \delta_{r,-s}\alpha^{(r)}_i \\
&C^{rs}_t = \begin{cases} \varepsilon(r,s) & \text{if } \alpha^{(r)} + \alpha^{(s)} = \alpha^{(t)} \text{ is a root} \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
\qquad (2.2)
$$

The indices i, j, etc $= 1, \ldots, l$ and the indices r, s, etc $= \pm 1, \ldots, \pm N_G$ ($N_G = (d_G - l)/2$ with $l = \text{rank G}$ and $d_G = \dim G$) label the Cartan subalgebra H of G and the coset G/H respectively, and indices a, b, etc label the generators of G. The roots of G, correspondingly, are labelled as follows:

$$
\begin{aligned}
&\text{positive roots} \qquad \alpha^{(1)}, \ldots, \alpha^{(N_G)} \\
&\text{negative roots} \qquad \alpha^{(-1)}, \ldots, \alpha^{(-N_G)}
\end{aligned}
\qquad (2.3)
$$

so that $\alpha^{(-s)} = -\alpha^{(s)}$.

The advantage of using the Cartan–Weyl bases is that it leads to the one-to-one correspondence between the modes and the root system of the affine algebra $\hat{G}$ [8] and, as we shall see later on, it makes the derivation of the partial differential equations for the characters easier.

The ground states (tachyon states) of the theory are the highest weight vectors of the integrable highest weight representation $L(\Lambda)$ (with the highest weights Λ) of the affine algebra $\hat{G}$,

$$J_n^a|\Lambda\rangle = 0 \qquad n>0$$
$$J_0^i|\Lambda\rangle = \Lambda_i|\Lambda\rangle \qquad \Lambda_i \in Z_+ \tag{2.4}$$
$$J_0^s|\Lambda\rangle = 0 \qquad s>0$$

and the Hilbert space of the theory decomposes into a finite sum of the integrable highest weight representations of the given level. The energy–momentum tensor is given by the Sugawara construction:

$$T(z) = \frac{1}{2(C_A+K)} :J_a(z)J^a(z): \tag{2.5}$$

where C_A is the second Casimir of the adjoint representation.

In order to derive the partial differential equations for the character of $\angle(\Lambda)$ we need to have the current Ward identities [5, 6]

$$\begin{aligned}
&\langle J^a(z)J^{b_1}(w_1)\dots J^{b_n}(w_n)\rangle - \langle J^a(z)\rangle\langle J^{b_1}(w_1)\dots J^{b_n}(w_n)\rangle \\
&\quad = \sum_{i=1}^{n} K\partial_{w_i}G^{ab_i}(z,w_i)\langle J^{b_1}(w_1)\dots J^{b_{i-1}}(w_{i-1})J^{b_{i+1}}(w_{i+1})\dots J^{b_n}(w_n)\rangle \\
&\qquad + \sum_{i=1}^{n} G^a_c(z,w_i)C^{cb_i}_d\langle J^{b_1}(w_1)\dots J^{b_{i-1}}(w_{i-1})J^d(w_i)J^{b_{i+1}}(w_{i+1})\dots J^{b_n}(w_n)\rangle \\
&\qquad + \mathscr{L}_a\langle J^{b_1}(w_1)\dots J^{b_n}(w_n)\rangle
\end{aligned} \tag{2.6}$$

where

$$G(z,w) = \frac{2\pi i}{u-1} + 2\pi i \sum_{n=1}^{\infty}\left(\frac{u^{-n}}{e^{-\gamma}-q^n} - \frac{u^n}{e^{\gamma}-q^n}\right)q^n$$
$$u = e^{2\pi i(z-w)} \qquad q = e^{2\pi i\tau} \tag{2.7}$$
$$\gamma = 2\pi i\xi^i t^i \qquad t^i \in \text{adjoint representation of G} \tag{2.8}$$

and $\mathscr{L}_a$ denotes the Lie derivative on the group manifold G along the left-invariant Killing vector $e^a_\angle$. In equation (2.6), the correlation functions are defined by

$$\langle J^{a_1}(z_1)\dots J^{a_n}(z_n)\rangle = Z^{-1}(\tau,\xi)\,\mathrm{Tr}(q^{\angle_0 - C_G/24}\,e^{\gamma}J^{a_1}(z_1)\dots J^{a_n}(z_n)) \tag{2.9}$$
$$Z(\tau,\xi) \equiv Z(\tau,\xi^1,\dots,\xi^l) = \mathrm{Tr}(q^{\angle_0 - C_G/24}\,e^{\gamma}) \tag{2.10}$$

with $C_G = K\,d_G/C_A + K$.

We now derive the heat equation for characters $Z(\tau,\xi)$ (precisely speaking, $Z = q^{S_\Lambda}\,\mathrm{Ch}_{\angle(\Lambda)}(\tau,\xi,0)$, where

$$S_\Lambda = \frac{C_\Lambda}{C_A+K} - \frac{C_G}{24}$$

is the dimension of the field ϕ_Λ less the trace anomaly and $\mathrm{Ch}_{\angle(\Lambda)}(\tau,\xi,t)$ is the character of the representation $\angle(\Lambda)$) by using equations (2.5)–(2.6) and the energy–momentum tensor Ward identity

$$2\pi i\frac{\partial Z(\tau,\xi)}{\partial\tau} = \langle T(\xi)\rangle. \tag{2.11}$$

Setting $n=1$ in equation (2.6) and using equation (2.2), we find

$$\langle J^a(z)J^i(w)\rangle = K\delta^{ai}\left(\frac{1}{(z-w)^2}-2\eta_1\right)+\langle J_0^a J_0^i\rangle+\mathrm{O}(z-w) \tag{2.12a}$$

$$\langle J^a(z)J^s(w)\rangle = \delta^{a,-s}\left[K\left(\frac{1}{(z-w)^2}+\eta_1-\frac{1}{2}\frac{1}{\theta_1}\frac{\partial^2\theta_1}{\partial x_s^2}\right)+\left(-\frac{1}{z-w}+\frac{1}{\theta_1}\frac{\partial\theta_1}{\partial x_s}\right)\alpha_i^{(s)}\langle J_0^i\rangle\right] + \mathrm{O}(z-w) \tag{2.12b}$$

where

$$\eta_1 = 2\pi\mathrm{i}\frac{\mathrm{d}\ln\eta}{\mathrm{d}\tau}$$

($\eta(\tau)=q^{1/24}\Pi_{n=1}(1-q^n)$ is the Dedekind function), $x_s=\alpha_i^{(s)}\xi^i$, $\theta_1=\theta_1(\tau, x_s)$ is the standard elliptic theta function and we have used [6]

$$\langle J_0^a\phi_1\ldots\phi_n\rangle - \langle J_0^a\rangle\langle\phi_1\ldots\phi_n\rangle = \mathscr{L}a\langle\phi_1\ldots\phi_n\rangle \tag{2.13}$$

with ϕ_i being the field or the current. From equations (2.5), (2.11), (2.12), and $\langle J_0^i\rangle = \partial\ln Z/\partial\xi^i$, one obtains the following heat equation

$$2(C_{\mathrm{A}}+K)2\pi\mathrm{i}\frac{\partial\ln Z}{\partial\tau} = (d_{\mathrm{G}}-3l)K\eta_1 - K\sum_{s=1}^{N_{\mathrm{G}}}\frac{1}{\theta_1}\frac{\partial^2\theta_1}{\partial x_s^2}+2\sum_{s=1}^{N_{\mathrm{G}}}\frac{\partial\ln\theta_1}{\partial x_s}\alpha_i^{(s)}\frac{\partial\ln Z}{\partial\xi^i}+\frac{1}{Z}\sum_{i=1}^{l}\frac{\partial^2 Z}{\partial\xi^{i2}}. \tag{2.14}$$

The heat equation has been derived in [6] where it possesses a different appearance. Dependence on the roots of an algebra in equation (2.14) expresses dependence on the algebra explicitly, which is the consequence of using Cartan-Weyl bases.

It is clear that equation (2 14) alone cannot completely determine the character Z. We need to have more equations in order to completely determine Z. We shall show that such equations follow from null vector states of the theory, which are due to the current symmetry, and the current Ward identities, equation (2.6).

From the Kac-Moody algebra (2.1), we know that, for each given value of s and m, one has the following SU_2 algebra

$$[J_m^{-s}, J_{-m}^{s}] = P_m^s \tag{2.15a}$$

$$[P_m^s, J_{\mp m}^{\pm s}] = \mp J_{\mp m}^{\pm s} \tag{2.15b}$$

with

$$P_m^s = \frac{2}{|\alpha^{(s)}|^2}(Km-\alpha_i^{(s)}J_0^i). \tag{2.16}$$

For the highest weight state $|\Lambda\rangle$, equation (2.4) leads to

$$P_m^s|\Lambda\rangle = \frac{2}{|\alpha^{(s)}|^2}(Km-(\alpha^{(s)},\Lambda))|\Lambda\rangle \equiv M_m^s(K,\Lambda)|\Lambda\rangle. \tag{2.17}$$

It is straightforward from the integrable property of the representation and equations (2.1) and (2.4) to prove that

$$|\chi^s_m(K,\Lambda)\rangle=(J^s_{-m})^{M^s_m(K,\Lambda)+1}|\Lambda\rangle=0 \tag{2.18}$$

is a null vector state. (The case of $m=1$ and $\alpha^{(s)}=$ the highest root of G is discussed in [3].) For our purpose we take $m=1$ and $|\Lambda\rangle=|0\rangle$, singlet, so that

$$M^s_K\equiv M^s_1(K,0)=\frac{2K}{|\alpha^{(s)}|^2} \tag{2.19}$$

and we have null vector states

$$|\chi^s_1(K,0)\rangle=(J^s_{-1})^{M^s_K+1}|0\rangle=0$$

or null vector fields

$$\chi^s_K=(J^s_{-1})^{M^s_K+1}I=0. \tag{2.20}$$

Now we would like to derive partial differential equations for characters of $\hat{G}$ by using null vector fields and current Ward identities. Let us consider the case of level one ($K=1$) first. Then from equations (2.19) and (2.20) we have

(1) Simply-laced algebras (i.e. A_l, D_l and E_l)

$$\chi^s_1=(J^s_{-1})^2=0 \tag{2.21}$$

(2) B_l, C_l and F_4

$$\chi^s_1=(J^s_{-1})^2=0 \qquad \text{for } \alpha^{(s)}\in\Delta_l \tag{2.22}$$

$$\chi^s_1=(J^s_{-1})^3=0 \qquad \text{for } \alpha^{(s)}\in\Delta_s \tag{2.23}$$

(3) G_2

$$\chi^s_1=(J^s_{-1})^2=0 \qquad \text{for } \alpha^{(s)}\in\Delta_l \tag{2.24}$$

$$\chi^s_1=(J^s_{-1})^4=0 \qquad \text{for } \alpha^{(s)}\in\Delta_s \tag{2.25}$$

where Δ_l (Δ_s) is the set of the long (short) roots of the algebras and the normalization of a root is the same as that in Kac's book [9], i.e.

$$|\alpha|^2=2\kappa \qquad \text{for } \alpha\in\Delta_l$$

$$|\alpha|^2=2\kappa/s \qquad s=\max_{a_{ij}\neq 0} a_{ji}/a_{ij} \qquad \text{for } \alpha\in\Delta_s$$

where κ is the order of automorphism of the Dynkin diagram of G and a_{ij} is the element of the generalized Cartan matrix of the affine Lie algebra $\hat{G}$ (We discriminate between affine and finite algebras with a hat and without a hat respectively in this paper. For example, $\hat{\alpha}$ and α denote a root of affine and finite algebras respectively. However, when there can be no confusion we shall omit the hat for the sake of simplicity.)

We use the global G symmetry in order to obtain partial differential equations from null vectors

$$g\chi^s_1 g^{-1}=0 \qquad g\in G. \tag{2.26}$$

A little calculation shows

$$2\alpha^{(s)}_i\alpha^{(s)}_j\langle J^i_{-1}J^j_{-1}\rangle-|\alpha^{(s)}|^2\langle J^s_{-1}J^{-s}_{-1}+J^{-s}_{-1}J^s_{-1}\rangle=0 \text{ for } \chi^s_1=(J^s_{-1})^2 \tag{2.27}$$

$$|\alpha^{(s)}|^2\alpha^{(s)}_i\langle J^i_{-1}(J^s_{-1}J^{-s}_{-1}+J^{-s}_{-1}J^s_{-1}))+(J^s_{-1}J^{-s}_{-1}+J^{-s}_{-1}J^s_{-1})J^i_{-1}+J^s_{-1}J^i_{-1}J^{-s}_{-1}$$
$$+J^{-s}_{-1}J^i_{-1}J^s_{-1}\rangle-2\alpha^{(s)}_i\alpha^{(s)}_j\alpha^{(s)}_k\langle J^i_{-1}J^j_{-1}J^k_{-1}\rangle=0 \text{ for } \chi^s_1=(J^s_{-1})^3. \tag{2.28}$$

Recall that

$$\langle J^a_{-1}J^b_{-1}J^c_{-1}(z)\rangle$$
$$=\oint_{w,v,z} \mathrm{d}y(y-z)^{-1}\oint_{v,z} \mathrm{d}w(w-z)^{-1}\oint_{z} \mathrm{d}v(v-z)^{-1}\langle J^a(y)J^b(w)J^c(v)\rangle.$$

By using the current Ward identities (2.6) and equations (2.27) to (2.29), we obtain the following partial differential equations for the characters:

(1) Simply laced algebras

$$D_s^{(2)}Z=0 \qquad s=1,\ldots,N_G \tag{2.30}$$

$$D_s^{(2)}\equiv\frac{1}{\theta_1}\frac{\partial^2\theta_1}{\partial x_s^2}-6\eta_1-2\frac{\partial\ln\theta_1}{\partial x_s}\alpha_i^{(s)}\frac{\partial}{\partial\xi^i}+\alpha_i^{(s)}\alpha_j^{(s)}\frac{\partial^2}{\partial\xi^i\partial\xi^j} \tag{2.31}$$

(2) B_l, C_l, F_4

$$D_s^{(2)}Z=0 \qquad \text{for } \alpha^{(s)}\in\Delta_l \tag{2.32a}$$

$$D_s^{(3)}Z=0 \qquad \text{for } \alpha^{(s)}\in\Delta_s \tag{2.32b}$$

$$D_s^{(3)}\equiv\frac{1}{2}\frac{1}{\theta_1}\frac{\partial^3\theta_1}{\partial x_s^3}-\frac{3}{2}\frac{\partial\ln\theta_1}{\partial x_s}\frac{1}{\theta_1}\frac{\partial^2\theta_1}{\partial x_s^2}+6\eta_1\frac{\partial\ln\theta_1}{\partial x_s}+3\left[\left(\frac{\partial\ln\theta_1}{\partial x_s}\right)^2-2\eta_1\right]\alpha_i^{(s)}\frac{\partial}{\partial\xi^i}$$
$$-3\frac{\partial\ln\theta_1}{\partial x_s}\alpha_i^{(s)}\alpha_j^{(s)}\frac{\partial^2}{\partial\xi^i\partial\xi^j}+\alpha_i^{(s)}\alpha_j^{(s)}\alpha_k^{(s)}\frac{\partial^3}{\partial\xi^i\partial\xi^j\partial\xi^k}. \tag{2.33}$$

It is easy to see from these results that the number of the modes contained in a null vector is equal to the order of the partial differential equation derived from the null vector and this fact is essentially due to the definition of the action of the zero modes J_0^a. For $G=G_2$ fourth order partial differential equations will be similarly derived and we shall not discuss G_2 hereafter for the sake of simplicity.

The same procedures can be applied to higher level cases and the higher-order partial differential equations will be obtained. For instance, at level $K=2$, we have third-order partial differential equations for simply-laced algebras and third- and fifth-order partial differential equations for B_l, C_l and F_4. The reason for the increase in the order with increasing K is that the number of modes (J^s_{-1}) contained in the null vectors increases with increasing K (see equations (2.19) and (2.20)).

3. Solutions of partial differential equations for $K=1$

The partial differential equations (2.30) for simply-laced algebras or (2.32) for B_l, C_l and F_4 are not independent because the number of the equations is $N_G=(d_G-l)/2$, the number of the positive roots of the algebra G (it is easy to see that the substitution $\alpha^{(s)}\to\alpha^{(-s)}$ leads to the same equation), and it exceeds the number of variables, l, except in the case of A_1. The number of independent partial differential equations depends on the structure of $\hat{G}$. Given $\hat{G}$, one can find out the independent ones from the partial differential equations.

We now combine the heat equation and the set of partial differential equations originated from null vectors and find the solutions of these equations for simply laced algebras and B_l.

3.1. Simply laced algebras

For $K=1$, we have N_G partial differential equations (2.30) Summing these N_G equations, we obtain

$$\sum_{s=1}^{N_G}\left[\frac{1}{\theta_1}\frac{\partial^2\theta_1}{\partial x_s^2}-2\frac{\partial \ln\theta_1}{\partial x_s}\alpha_i^{(s)}\frac{1}{Z}\frac{\partial Z}{\partial \xi^i}\right]-3(d_G-l)\eta_1+C_A\frac{1}{Z}\sum_{i=1}^{l}\frac{\partial^2 Z}{\partial \xi^{i^2}}=0 \tag{3.1}$$

where we have used

$$\sum_{s=1}^{N_G}\alpha_i^{(s)}\alpha_j^{(s)}=C_A\delta_{ij} \tag{3.2}$$

Taking $K=1$ in equation (2.14), we have

$$(C_A+1)4\pi i\frac{\partial \ln Z}{\partial\tau}$$
$$=(d_G-3l)\eta_1-\sum_{s=1}^{N_G}\left[\frac{1}{\theta_1}\frac{\partial^2\theta_1}{\partial x_s^2}-2\frac{\partial\ln\theta_1}{\partial x_s}\alpha_i^{(s)}\frac{1}{Z}\frac{\partial Z}{\partial\xi^i}\right]+\frac{1}{Z}\sum_{i=1}^{l}\frac{\partial^2 Z}{\partial\xi^{i^2}}. \tag{3.3}$$

Substituting equation (3.1) into equation (3.3), we obtain

$$(C_A+1)4\pi i\frac{\partial\ln Z}{\partial\tau}=-2d_G\eta_1+(C_A+1)\frac{1}{Z}\sum_{i=1}^{l}\frac{\partial^2 Z}{\partial\xi^{i^2}}. \tag{3.4}$$

Note that for simply laced algebras,

$$d_G=l(C_A+1) \tag{3.5}$$

From equations (3.4) and (3.5), we have finally

$$4\pi i\frac{\partial(\eta^l Z)}{\partial\tau}=\sum_{i=1}^{l}\frac{\partial^2(\eta^l Z)}{\partial\xi^{i^2}} \tag{3.6}$$

Thus we obtain the following solutions.

$$Z=\frac{\theta_\Lambda(\tau,\xi,0)}{\eta^l(\tau)} \tag{3.7}$$

where

$$\theta_\Lambda(\tau,\xi,t)=e^{-2\pi i t}\sum_{\alpha\in M-\Lambda}\exp[\pi i\tau|\alpha|^2-2\pi i(\alpha_1\xi^1+\ldots+\alpha_l\xi^l)] \tag{3.8}$$

is the classical theta function of degree 1 (with characteristic Λ) [9]. These are exactly the characters for $G=A_l^{(1)}$, $D_l^{(1)}$ and $E_l^{(1)}$ which have been given by Kac [9].

3.2. B_l

Because $\alpha_i^{(s)}=\delta_{si}$ for $\alpha^{(s)}\in\Delta_s$, the equations (2.32*b*) decouple, i.e one has the l independent equations each of which contains only the partial differentials for one variable:

$$\left(\frac{1}{2}\frac{1}{\theta_1}\frac{\partial^2\theta_1}{\partial\xi^{i^2}}-\frac{3}{2}\frac{\partial\ln\theta_1}{\partial\xi^i}\frac{1}{\theta_1}\frac{\partial^2\theta_1}{\partial\xi^{i^2}}+6\eta_1\frac{\partial\ln\theta_1}{\partial\xi^i}\right)Z+3\left[\left(\frac{\partial\ln\theta_1}{\partial\xi^i}\right)^2-2\eta_1\right]\frac{\partial Z}{\partial\xi^i}$$
$$-3\frac{\partial\ln\theta_1}{\partial\xi^i}\frac{\partial^2 Z}{\partial\xi^{i^2}}+\frac{\partial^3 Z}{\partial\xi^{i^3}}=0\qquad i=1,\ldots,l. \tag{3.9}$$

(Note that now $x_s = \alpha_i^{(s)} \xi^i = \xi^s$, $s = 1, \ldots, l$.) Therefore the solutions assume the factorized form

$$Z(\tau, \xi) = C(\tau) \prod_{i=1}^{l} f_i(\tau, \xi^i). \tag{3.10}$$

Substituting equation (3.10) into equation (3.9), we obtain the same l third order differential equations for f_i and the solutions of the equation are

$$f_i(\tau, \xi^i) = \theta_m(\tau, \xi^i) \qquad m = 2, 3, 4 \tag{3.11}$$

where θ_m is the usual theta function on the torus [10]. The unknown function $C(\tau)$ of equation (3.10) can be determined by the heat equation

$$(C_A + 1) 4\pi i \frac{\partial \ln Z}{\partial \tau} = \sum_{i=1}^{l} \left[-4\pi i \frac{\partial \ln \theta_1(\tau, \xi^i)}{\partial \tau} - 2(C_A - 1)\eta_1 + 2 \frac{\partial \ln \theta_1(\tau, \xi^i)}{\partial \xi^i} \frac{\partial \ln Z}{\partial \xi^i} + C_A \frac{1}{Z} \frac{\partial^2 Z}{\partial \xi^{i2}} \right] \tag{3.12}$$

where we have used

$$\frac{\partial^2 \theta_1(\tau, \zeta)}{\partial \zeta^2} = 4\pi i \frac{\partial \theta_1(\tau, \zeta)}{\partial \tau}. \tag{3.13}$$

From equations (3.10), (3.11) and (3.12) we finally find the solutions as follows

$$Z_m = C_m(\tau) \prod_{i=1}^{l} \theta_m(\tau, \xi^i) \qquad m = 2, 3, 4 \tag{3.14}$$

with

$$\begin{aligned} C_2(\tau) &= \eta^{-(l+1)}(\tau)\eta(2\tau) \\ C_3(\tau) &= \eta^{-l}(\tau/2)\eta^{2-l}(\tau)\eta^{-1}(2\tau) \\ C_4(\tau) &= \eta^{-(l+1)}(\tau)\eta(\tau/2). \end{aligned} \tag{3.15}$$

It is easy to check that the solutions satisfy equations (2.32*a*), as they should. By linear combination, the solutions can be written in a familiar form, i.e

$$\begin{aligned} Z_{\Lambda_l} &= C^{\Lambda_l}_{\Lambda_l}(\tau)\theta_{\Lambda_l}(\tau, \xi) \\ Z_{\Lambda_0} &= C^{\Lambda_0}_{\Lambda_0}(\tau)\theta_{\Lambda_0}(\tau, \xi) + C^{\Lambda_0}_{\Lambda_1}(\tau)\theta_{\Lambda_1}(\tau, \xi) \\ Z_{\Lambda_1} &= C^{\Lambda_1}_{\Lambda_0}\theta_{\Lambda_0} + C^{\Lambda_1}_{\Lambda_1}\theta_{\Lambda_1} \end{aligned} \tag{3.16}$$

$$\begin{aligned} &C^{\Lambda_1}_{\Lambda_1} - C^{\Lambda_1}_{\Lambda_0} = C_3 \qquad C^{\Lambda_1}_{\Lambda_1} + C^{\Lambda_1}_{\Lambda_0} = C_4 \qquad C^{\Lambda_l}_{\Lambda_l} = C_2 \\ &C^{\Lambda_0}_{\Lambda_0} = C^{\Lambda_1}_{\Lambda_1} \qquad C^{\Lambda_1}_{\Lambda_0} = C^{\Lambda_0}_{\Lambda_1} \end{aligned} \tag{3.17}$$

where Λ_l, Λ_0 and Λ_1 are all integrable representations of level one of $\hat{B}_l$. Equation (3.16) is identical to the Weyl–Kac character formula at $K = 1$ for $\hat{B}_l$.

4. Calculation of string functions

The algebraic methods and some results of calculating string functions have been given by Kac and Peterson [11]. Here we show by means of some examples how to calculate string functions by using analytical methods (i.e. using the partial differential equation for characters).

Now our starting point is the Weyl–Kac character formula

$$Z_\Lambda(\tau, \xi, 0) = \sum_{\lambda \in P_+^K} C_\lambda^\Lambda(\tau)\theta_\lambda(\tau, \xi) \tag{4.1}$$

where θ_λ is the classical theta function of degree K (see equation (3.8) for $K=1$) [9] and our purpose is to find out the unknown string functions $C_\lambda^\Lambda(\tau)$. Let us denote equation (2.14) by

$$(C_A+K)4\pi i\frac{\partial Z}{\partial \tau} = DZ \tag{4.2}$$

$$D = \sum_{i=1}^{l}\frac{\partial^2}{\partial \xi^{i^2}} + 2\sum_{s=1}^{N_G}\frac{\partial \ln \theta_1}{\partial x_s}\alpha_i^{(s)}\frac{\partial}{\partial \xi^i} + (d_G - 3l)K\eta_1 - K\sum_{s=1}^{N_G}\frac{1}{\theta_1}\frac{\partial^2\theta_1}{\partial x_s^2}. \tag{4.3}$$

Substituting $Z = C(\tau)f(\tau, \xi)$ into equation (4.2), we have

$$(C_A+K)4\pi i\frac{d \ln C}{d\tau} = \frac{1}{f}\left(D - (C_A+K)4\pi i\frac{\partial}{\partial \tau}\right)f. \tag{4.4}$$

Because the left-hand side of equation (4.4) is independent of the variables $\xi^1, \ldots, \xi^l$, the right-hand side of equation (4.4) should also be so. Therefore, we can calculate it for any values of $\xi^1, \ldots, \xi^l$ and the easy way is to calculate it at $\xi^1 = \ldots = \xi^l = 0$ (we shall write $\xi = 0$ instead of $\xi^1 = \ldots = \xi^l = 0$ for the sake of simplicity). Thus we have

$$\begin{aligned} Df|_{\xi=0} = {} & \sum_{i=1}^{l}\frac{\partial^2 f}{\partial \xi^{i^2}}\bigg|_{\xi=0} + 2\frac{\partial \ln \theta_1(\tau, x)}{\partial x}\bigg|_{x=0}\frac{\partial f}{\partial \xi^i}\bigg|_{\xi=0}P_i^G \\ & + \left[(d_G - 3l)K\eta_1 - K4\pi i\frac{\partial \ln \theta_1(\tau, 0)}{\partial \tau}N_G\right]f(\tau, 0) \end{aligned} \tag{4.5}$$

where

$$P_i^G = \sum_{s=1}^{N_G}\alpha_i^{(s)}. \tag{4.6}$$

We now turn to some examples.

4.1. $B_l^{(1)}$

The spinor representation of level one of $B_l^{(1)}$ is labelled by the highest weight $\hat{\Lambda}_l = (0, \Lambda_l)$ with

$$\Lambda_l = (0^{l-1}1) = \sum_{j=1}^{l}\frac{j}{2}\alpha^{(j)} \tag{4.7}$$

where $\alpha^{(j)}$ is the simple root of B_l. From equation (3.8) one has

$$\theta_{\Lambda_l} = \prod_{j=1}^{l}\theta_2(\tau, \xi^j). \tag{4.8}$$

Taking $K=1$ and $f=\theta_{\Lambda_l}$ in equations (4.4) and (4.5), a straightforward calculation shows the right-hand side of equation (4.4) is equal to

$$4\pi \mathrm{i}\left[-2l(l+1)\frac{\mathrm{d}\ln\eta(\tau)}{\mathrm{d}\tau}+2l\frac{\mathrm{d}\ln\eta(2\tau)}{\mathrm{d}\tau}\right] \tag{4.9}$$

Thus we find

$$C(\tau)=\eta^{-(l+1)}(\tau)\eta(2\tau) \tag{4.10}$$

up to a multiplicable constant, as expected.

4.2. $D_l^{(1)}$

Consider the fundamental representation $\angle(\hat{\Lambda}_1)$ of $K=1$ of $D_l^{(1)}$ with $\hat{\Lambda}_1=(010^{l-1})=(0,\Lambda_1)$, $\Lambda_1=(10^{l-1})=e_1$. From equation (3 8) we have

$$\theta_{\Lambda_1}=\frac{1}{2}\left(\prod_{i=1}^{l}\theta_3(\tau,\xi^i)-\prod_{i=1}^{l}\theta_4(\tau,\xi^i)\right). \tag{4.11}$$

The calculation will be very complicated if one substitutes directly $Z_{\Lambda_1}=C_{\Lambda_1}^{\Lambda_1}\theta_{\Lambda_1}$ into equation (4.2) since equation (4.11) is a non-factorized expression. Fortunately, because the operator D is a linear operator we can use the principle of superposition for solutions. Recall that

$$\begin{aligned} &Z_{\Lambda_0}=C_{\Lambda_0}^{\Lambda_0}(\tau)\theta_{\Lambda_0} \qquad \hat{\Lambda}_0=(1,0^l) \qquad \Lambda_0=(0^l)\\ &\theta_{\Lambda_0}=\frac{1}{2}\left(\prod_{i=1}^{l}\theta_3(\tau,\xi^i)+\prod_{i=1}^{l}\theta_4(\tau,\xi^i)\right)\end{aligned} \tag{4.12}$$

and $C_{\Lambda_0}^{\Lambda_0}=C_{\Lambda_1}^{\Lambda_1}$. From equations (4.11) and (4.12), one has

$$Z_{\Lambda_1}+Z_{\Lambda_0}=C_{\Lambda_1}^{\Lambda_1}(\tau)\prod_{i=1}^{l}\theta_3(\tau,\xi^i). \tag{4.13}$$

Substituting equation (4.13) into equation (4.2), one can easily find

$$C_{\Lambda_1}^{\Lambda_1}=\eta^{-l}(\tau) \tag{4.14}$$

as expected.

4.3. $C_3^{(1)}$

There are $(l+1)$ representations of level one of $C_l^{(1)}$ and the calculations of string functions for $C_l^{(1)}$ is still an open problem [9]. Here we report the results for $l=3$ and the calculation for arbitrary l is in progress.

The highest weights of the $K=1$ representations for $C_3^{(1)}$ are $\hat{\Lambda}_0$, $\hat{\Lambda}_1$, $\hat{\Lambda}_2$ and $\hat{\Lambda}_3$. As an example, let us focus our attention to $\angle(\hat{\Lambda}_3)$. According to the Weyl–Kac character formula (4.1), one has

$$Z_{\Lambda_3}=C_{\Lambda_1}^{\Lambda_3}\theta_{\Lambda_1}+C_{\Lambda_3}^{\Lambda_3}\theta_{\Lambda_3} \tag{4.15a}$$

$$=C_{\Lambda_1}^{\Lambda_3}(\theta_{\Lambda_1}+R(\tau)\theta_{\Lambda_3}) \tag{4.15b}$$

where θ_{Λ_1} and θ_{Λ_3} can be expressed as

$$\theta_{\Lambda_1}=\theta_e\left(\frac{\tau}{2},\frac{\xi^2}{\sqrt{2}}\right)\left(\theta_0\left(\frac{\tau}{2},\frac{\xi^1}{\sqrt{2}}\right)\theta_e\left(\frac{\tau}{2},\frac{\xi^3}{\sqrt{2}}\right)+\theta_e\left(\frac{\tau}{2},\frac{\xi^1}{\sqrt{2}}\right)\theta_0\left(\frac{\tau}{2},\frac{\xi^3}{\sqrt{2}}\right)\right) \tag{4.16}$$

$$\theta_{\Lambda_3}=\theta_0\left(\frac{\tau}{2},\frac{\xi^2}{\sqrt{2}}\right)\left(\theta_e\left(\frac{\tau}{2},\frac{\xi^1}{\sqrt{2}}\right)\theta_e\left(\frac{\tau}{2},\frac{\xi^3}{\sqrt{2}}\right)+\theta_0\left(\frac{\tau}{2},\frac{\xi^1}{\sqrt{2}}\right)\theta_0\left(\frac{\tau}{2},\frac{\xi^3}{\sqrt{2}}\right)\right) \tag{4.17}$$

with

$$\theta_e=\tfrac{1}{2}(\theta_3+\theta_4) \tag{4.18a}$$

$$\theta_0=\tfrac{1}{2}(\theta_3-\theta_4). \tag{4.18b}$$

From equations (2.32*b*) and (4.15*b*), we obtain

$$R(\tau)=-\frac{D_s^{(3)}\theta_{\Lambda_1}}{D_s^{(3)}\theta_{\Lambda_3}}. \tag{4.19}$$

By using equations (2.33), (4.16), (4.17) and (4.19), it is straightforward to derive that

$$R(\tau)=\frac{\theta_e(0)(\theta_0''(0)\theta_e(0)-5\theta_e''(0)\theta_0(0))}{\theta_0''(0)(2\theta_0^2(0)+3\theta_e^2(0))-\theta_e''(0)\theta_e(0)\theta_0(0)} \tag{4.20}$$

where we write $\theta_e(0)$, $\theta_e''(0)$, etc instead of $\theta_e(\tau,0)$, $\partial^2\theta_e(\tau,\zeta)/\partial\zeta^2|_{\zeta=0}$, etc for the sake of simplicity. Setting $K=1$ and $f=\theta_{\Lambda_1}+R(\tau)\theta_{\Lambda_3}$ in equations (4.4) and (4.5), we find by a straightforward calculation

$$C_{\Lambda_1}^{\Lambda_3}=\eta^{-21/5}\phi^{2/5}\exp\left\{-\frac{7}{5}\int\frac{\mathrm{d}R}{\mathrm{d}\tau}\frac{\theta_{\Lambda_3}(0)}{\phi}\,\mathrm{d}\tau\right\} \tag{4.21}$$

where

$$\phi=\theta_{\Lambda_1}(0)+R\theta_{\Lambda_3}(0)=(2+R)\theta_e^2(0)\theta_0(0)+R\theta_0^3(0). \tag{4.22}$$

Finally, according to the definition of R (see equation (4.15)), we have

$$C_{\Lambda_3}^{\Lambda_3}=RC_{\Lambda_1}^{\Lambda_3}. \tag{4.23}$$

5. Discussion

In summary, we have derived the partial differential equations for characters of Kac–Moody algebras by using current and conformal Ward identities, the Sugawara construction and the null vectors from the Kac-Moody algebras. For level-one representations of $A_l^{(1)}$, $D_l^{(1)}$, $E_l^{(1)}$ and $B_l^{(1)}$, we have given the solutions of the partial differential equations and consequently a complete 'physical' proof of the Weyl–Kac character formula in these cases. The partial differential equations for other affine algebras and higher levels can also be solved and we leave this to the future.

The number of the partial differential equations, in general, is larger than that of the independent variables. We know from the origin of these equations that they are all consistent with each other. But, unfortunately, there is no general way by which one can determine the number of the independent partial differential equations. The number depends on the specific algebra. Given an algebra, one can find the independent ones from equation (2.30) or (2.32) and solve them so that all the independent solutions can be found.

For higher levels, we need to solve higher-order partial differential equations. Although there is no difficulty, in principle, in solving them, this will be technically complicated to achieve in fact. Alternatively, we can use the method of decomposing a direct product representation and obtaining the higher level characters from the lower level ones, as discussed in [7] for $\hat{G}=A_1^{(1)}$. Nevertheless, this means the proof will not be completely analytic.

A generalization to semi-simple and twisted affine algebras is straightforward. By using the supercurrent and superconformal Ward identities on the supertorus [12], these methods can be generalized to supercharacters. As for generalization to higher genus ($g>1$) Riemann surfaces, there is much work to be done.

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References

[1] Belavin A A, Polyakov A M and Zamolodchikov A B 1984 *Nucl Phys.* B **241** 333
Friedan D, Qiu Z and Shenker S H 1984 *Phys. Rev Lett* **52** 1575
Goddard P, Kent A and Olive D 1985 *Phys Lett* **152B** 88
Kac V G and Todorov T 1985 *Comm Math. Phys* **102** 337
Friedan D, Martinec E and Shenker S 1986 *Nucl Phys* B **271** 93
Friedan D and Shenker S 1987 *Nucl. Phys.* B **281** 509
Verlinde E 1988 *Nucl. Phys.* B **300** 360
[2] Wess J and Zumino B 1971 *Phys Lett* **37B** 95
Novikov S P 1981 *Sov Math Dokl* **24** 222
Witten E 1984 *Comm. Math Phys* **92** 455
Polyakov A M and Wiegman P B 1983 *Phys Lett.* **141B** 223
[3] Gepner D and Witten E 1986 *Nucl Phys.* B **278** 493
[4] Sonoda H 1987 *Nucl. Phys* B **281** 546
[5] Eguchi T and Ooguri H 1987 *Nucl Phys.* B **282** 308
[6] Bernard D 1988 *Nucl. Phys* B **303** 77
[7] Eguchi T and Ooguri H 1989 *Nucl Phys* B **313** 492
[8] Goddard P and Olive D 1986 *Mod Phys* A **1** 303
[9] Kac V G 1983 *Infinite Dimensional Lie Algebras* (Boston: Birkhauser)
[10] Whittaker E T and Watson G N 1927 *A Course of Modern Analysis* (Cambridge: Cambridge University Press)
[11] Kac V G and Peterson D H 1984 *Adv. Math* **53** 125
[12] Grundberg J and Nakayama R 1988 *Nucl Phys* B **306** 497
Hamada K-Ji and Takao M 1988 *Preprint* KEK-TH-209
Kawamato N and Saeki Y 1989 *Preprint* KUNS 970
Huang C S, Xu K W and Zhao Z Y 1989 *Preprint* IC/292/89